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On non-Abelian expanding waves

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Abstract
We study non-Abelian expanding waves that can be radiated from various sources of Yang–Mills fields. We find a new class of exact wave solutions to the Yang–Mills equations. These solutions are constructed for any gauge group with a compact semi-simple Lie algebra and embrace asymmetrical cases of radiations of expanding waves. They can be regarded as a reasonable generalization of wave solutions of the Maxwell electrodynamics. It is of interest to apply the found solutions to detect cosmic sources of Yang–Mills fields. In the case of fields with $SU(2)$ symmetry this could be realized by observing the interaction of such sources' radiation with neutrinos.

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As is well known, in the case of an N-parameter gauge group Yang–Mills fields are described by the following equations outside their sources [1, 2]:

$$\partial_\mu F^{a,\mu\nu} + f_{abc} A^b_\mu F^{c,\mu\nu} = 0, \tag{1}$$

$$F^{a,\mu\nu} = \partial^\mu A^{a,\nu} - \partial^\nu A^{a,\mu} + f_{abc} A^{b,\mu} A^{c,\nu}, \tag{2}$$

where $\mu, \nu = 0, 1, 2, 3, a, b, c = 1, 2, \ldots, N, A^{a,\nu}$ and $F^{a,\mu\nu}$ are potentials and strengths of a Yang–Mills field, respectively, f_{klm} are the structure constants of an N-parameter gauge group and $\partial_\mu \equiv \partial/\partial x^\mu$, where x^μ are orthogonal spacetime coordinates of the Minkowski geometry.

One of the important problems is a search for exact wave solutions to the Yang–Mills equations (1)–(2). Plane wave solutions to equations (1)–(2) and their generalizations are most extensively studied and a number of interesting results are obtained [3–11]. Of special note is the class of non-Abelian plane waves found by Coleman [3]. This work opened up a fruitful way in the search for non-Abelian waves and gave impetus to a number of further investigations of the problem.

Our objective is to study expanding waves that can be radiated from sources of Yang–Mills fields.

Let us seek potentials $A^{a,\nu}$ of such wave solutions to the Yang–Mills equations in the following form:

$$A^{a,0} = u^a(y_0, y_1, y_2, y_3), \qquad A^{a,n} = (x^n/r)A^{a,0}, \qquad y_0 = x^0 - r, \qquad y_n = x^n,$$
$$n = 1, 2, 3, \qquad a = 1, 2, \ldots, N, \qquad r = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}, \tag{3}$$

where u^a are some functions of the wave phase $y_0 = x^0 - r$ and of the spatial coordinates $y_n = x^n$.

We will further consider gauge groups with compact semi-simple Lie algebras which have totally antisymmetric structure constants f_{abc} [2, 12]. Then, substituting expressions (3) into formula (2) for the field strengths $F^{a,\mu\nu}$, we readily find

$$F^{a,0n} = \partial u^a/\partial y_n, \qquad F^{a,in} = (1/r)(x^i \partial u^a/\partial y_n - x^n \partial u^a/\partial y_i),$$
$$y_n = x^n, \qquad i, n = 1, 2, 3. \tag{4}$$

Let us now substitute expressions (3) and (4) for $A^{a,\nu}$ and $F^{a,\mu\nu}$ into the Yang–Mills equation (1).

Then when the index $\nu = 0$ from (1) we obtain

$$\sum_{i=1}^{3} \left(\frac{\partial^2 u^a}{\partial y_i^2} - \frac{y_i}{r} \frac{\partial^2 u^a}{\partial y_0 \partial y_i} - \frac{y_i}{r} f_{abc} u^b \frac{\partial u^c}{\partial y_i} \right) = 0, \tag{5}$$

where $y_i = x^i$ and $y_0 = x^0 - r$. From this point on we shall label x^i by y_i when $i = 1, 2, 3$.

When the index $\nu = n = 1, 2, 3$ from equation (1) we obtain after reductions

$$\frac{y_n}{r} \sum_{i=1}^{3} \left(y_i \frac{\partial^2 u^a}{\partial y_0 \partial y_i} - r \frac{\partial^2 u^a}{\partial y_i^2} + \frac{y_i}{r} \frac{\partial u^a}{\partial y_i} + f_{abc} y_i u^b \frac{\partial u^c}{\partial y_i} \right) + \frac{\partial}{\partial y_n} \left(\sum_{i=1}^{3} y_i \frac{\partial u^a}{\partial y_i} \right) = 0. \tag{6}$$

As is readily seen, equations (5) and (6) are fulfilled when the functions u^a satisfy the following two equations which are sets of uncoupled linear partial differential equations:

$$\sum_{i=1}^{3} y_i \frac{\partial u^a}{\partial y_i} = 0, \qquad \sum_{i=1}^{3} \frac{\partial^2 u^a}{\partial y_i^2} = 0. \tag{7}$$

As follows from expressions (4) and the first equation in (7), the considered waves are transverse.

Let us turn to the first equation in (7) and show that it has the following solution:

$$u^a(y_0, y_1, y_2, y_3) = g^a(y_0, \xi_1, \xi_2, \xi_3), \qquad \xi_i = y_i/r, \qquad r = \sqrt{y_1^2 + y_2^2 + y_3^2}, \tag{8}$$

where g^a are arbitrary differentiable functions.

Actually, from (8) we derive

$$\frac{\partial u^a}{\partial y_i} = \frac{1}{r} \frac{\partial g^a}{\partial \xi_i} - \frac{y_i}{r^3} \sum_{n=1}^{3} y_n \frac{\partial g^a}{\partial \xi_n}, \qquad i = 1, 2, 3. \tag{9}$$

From (9) we get the identity $\sum_{i=1}^{3} y_i \partial u^a/\partial y_i \equiv 0$.

Therefore, formula (8) gives solutions to the first equation in (7).

From (8) we find

$$\frac{\partial g^a}{\partial y_i} = \frac{1}{r}\sum_{k=1}^{3}\frac{\partial g^a}{\partial \xi_k}(\delta_{ik} - \xi_i\xi_k), \qquad i = 1, 2, 3,$$

$$\xi_i = y_i/r, \qquad \delta_{ii} = 1, \qquad \delta_{ik} = 0 \qquad \text{when} \qquad k \neq i, \tag{10}$$

$$\frac{\partial^2 g^a}{\partial y_i^2} = \frac{1}{r^2}\sum_{k,n=1}^{3}\frac{\partial^2 g^a}{\partial \xi_k \partial \xi_n}(\delta_{ik} - \xi_i\xi_k)(\delta_{in} - \xi_i\xi_n) - \frac{1}{r^2}\sum_{k=1}^{3}\frac{\partial g^a}{\partial \xi_k}\left[\xi_k\left(1 - 3\xi_i^2\right) + 2\xi_i\delta_{ik}\right].$$

Let us substitute the functions $u^a = g^a(y_0, \xi_1, \xi_2, \xi_3)$ into the second part in (7). Then using formulae (10) and the equality $\xi_1^2 + \xi_2^2 + \xi_3^2 = 1$, we obtain

$$\sum_{i=1}^{3}\left[\left(1 - \xi_i^2\right)\frac{\partial^2 g^a}{\partial \xi_i^2} - 2\xi_i\frac{\partial g^a}{\partial \xi_i}\right] - \sum_{\substack{i,k=1\\ i\neq k}}^{3}\xi_i\xi_k\frac{\partial^2 g^a}{\partial \xi_i \partial \xi_k} = 0. \tag{11}$$

The arguments $\xi_i = y_i/r$ of the functions g^a are not independent, since $\xi_1^2+\xi_2^2+\xi_3^2 = 1$. That is why instead of ξ_1, ξ_2, ξ_3, we can choose two independent arguments related to them.

As will be seen below, it is convenient to choose the following two arguments θ and σ:

$$g^a(y_0, \xi_1, \xi_2, \xi_3) = h^a(y_0, \theta, \sigma), \qquad \theta = \frac{1}{2}\ln\left(\frac{1+\xi_1}{1-\xi_1}\right), \qquad \sigma = \arctan\left(\frac{\xi_2}{\xi_3}\right). \tag{12}$$

Then from (12) we find

$$\frac{\partial g^a}{\partial \xi_1} = \beta\frac{\partial h^a}{\partial \theta}, \qquad \frac{\partial g^a}{\partial \xi_2} = \gamma\xi_3\frac{\partial h^a}{\partial \sigma}, \qquad \frac{\partial g^a}{\partial \xi_3} = -\gamma\xi_2\frac{\partial h^a}{\partial \sigma}, \qquad \beta = \frac{1}{1-\xi_1^2},$$

$$\gamma = \frac{1}{\xi_2^2+\xi_3^2}, \qquad \frac{\partial^2 g^a}{\partial \xi_1^2} = \beta^2\left(\frac{\partial^2 h^a}{\partial \theta^2} + 2\xi_1\frac{\partial h^a}{\partial \theta}\right), \qquad \frac{\partial^2 g^a}{\partial \xi_2^2} = \gamma^2\left(\xi_3^2\frac{\partial^2 h^a}{\partial \sigma^2} - 2\xi_2\xi_3\frac{\partial h^a}{\partial \sigma}\right),$$

$$\frac{\partial^2 g^a}{\partial \xi_3^2} = \gamma^2\left(\xi_2^2\frac{\partial^2 h^a}{\partial \sigma^2} + 2\xi_2\xi_3\frac{\partial h^a}{\partial \sigma}\right), \qquad \frac{\partial^2 g^a}{\partial \xi_1 \partial \xi_2} = \beta\gamma\xi_3\frac{\partial^2 h^a}{\partial \theta \partial \sigma}, \tag{13}$$

$$\frac{\partial^2 g^a}{\partial \xi_1 \partial \xi_3} = -\beta\gamma\xi_2\frac{\partial^2 h^a}{\partial \theta \partial \sigma}, \qquad \frac{\partial^2 g^a}{\partial \xi_2 \partial \xi_3} = -\gamma^2\left(\xi_2\xi_3\frac{\partial^2 h^a}{\partial \sigma^2} + \left(\xi_3^2 - \xi_2^2\right)\frac{\partial h^a}{\partial \sigma}\right).$$

Let us substitute expressions (13) into equation (11). Then after reductions we obtain

$$\frac{1}{1-\xi_1^2}\frac{\partial^2 h^a}{\partial \theta^2} + \frac{1}{\xi_2^2+\xi_3^2}\frac{\partial^2 h^a}{\partial \sigma^2} = 0. \tag{14}$$

Since the variables $\xi_i = y_i/r$ satisfy the equality $\xi_2^2 + \xi_3^2 = 1 - \xi_1^2$, from (14) we come to the Laplace equation

$$\frac{\partial^2 h^a}{\partial \theta^2} + \frac{\partial^2 h^a}{\partial \sigma^2} = 0, \tag{15}$$

where, as is seen from (12), $-\infty < \theta < \infty$, $-\pi/2 \leqslant \sigma \leqslant \pi/2$.

Therefore, the problem under consideration is reduced to the following. We should find functions $h^a(y_0, \theta, \sigma)$ that are harmonic with respect to the arguments θ and σ in the band $-\pi/2 \leqslant \sigma \leqslant \pi/2$.

Let us denote

$$h_+^a(y_0, \theta) = h^a(y_0, \theta, \pi/2), \qquad h_-^a(y_0, \theta) = h^a(y_0, \theta, -\pi/2). \tag{16}$$

Then we have the classical problem of finding harmonic functions h^a in the band $-\infty < \theta < \infty$, $-\pi/2 \leqslant \sigma \leqslant \pi/2$ that take on arbitrary values h^a_+ and h^a_- when $\sigma = \pm\pi/2$. Its solution is described by the well-known Palatini formula [13, 14] which acquires the form

$$h^a(y_0, \theta, \sigma) = \mathrm{Re}\,\varphi^a(y_0, z), \qquad z = \theta + \mathrm{i}\sigma,$$
$$\varphi^a(y_0, z) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{h^a_+(y_0, s) + h^a_-(y_0, s)}{\cosh(s - z)}\,\mathrm{d}s - \frac{\mathrm{i}}{2\pi}\sinh z\int_{-\infty}^{\infty}\frac{h^a_+(y_0, s) - h^a_-(y_0, s)}{\cosh s\cosh(s - z)}\,\mathrm{d}s, \tag{17}$$

where the sought functions h^a are the real parts of the functions φ^a which are analytic functions of the complex argument $z = \theta + \mathrm{i}\sigma$ in the considered domain.

Using formulae (3), (8), (12) and (17), we find the field potentials $A^{a,\nu}$:

$$A^{a,0} = h^a\left(x^0 - r, \frac{1}{2}\ln\frac{r + x^1}{r - x^1}, \arctan\frac{x^2}{x^3}\right), \qquad A^{a,n} = (x^n/r)A^{a,0},$$
$$n = 1, 2.3, \qquad a = 1, 2, \ldots, N, \qquad r = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}. \tag{18}$$

The obtained formulae (3), (4), (17) and (18) give the sought expanding wave solutions to the Yang–Mills equations (1)–(2). These solutions are constructed for any gauge group with a compact semi-simple Lie algebra and embrace asymmetrical cases of radiations of expanding waves. They generalize wave solutions of the Maxwell electrodynamics and are determined by $2N$ arbitrary functions $h^a_+(y_0, \theta)$ and $h^a_-(y_0, \theta)$.

As an example, let us consider the particular case

$$h^a_\pm(y_0, \theta) = \frac{k^a_1(y_0)\sinh\theta \pm k^a_2(y_0)}{\cosh\theta}, \tag{19}$$

where k^a_1 and k^a_2 are some functions of the wave phase $y_0 = x^0 - r$.

Then from (17) we get

$$\varphi^a(y_0, z) = \frac{k^a_1(y_0)}{\pi}\int_{-\infty}^{\infty}\frac{\tanh s}{\cosh(s - z)}\,\mathrm{d}s - \frac{\mathrm{i}k^a_2(y_0)\sinh z}{\pi}\int_{-\infty}^{\infty}\frac{1}{\cosh^2 s\cosh(s - z)}\,\mathrm{d}s. \tag{20}$$

Let us denote

$$p = \exp s, \qquad q = \exp z. \tag{21}$$

Then the two integrals in (20) can be represented as

$$\int\frac{\tanh s}{\cosh(s - z)}\,\mathrm{d}s = 2q\int\frac{(p^2 - 1)\mathrm{d}p}{(p^2 + 1)(p^2 + q^2)},$$
$$\int\frac{1}{\cosh^2 s\cosh(s - z)}\,\mathrm{d}s = 8q\int\frac{p^2\mathrm{d}p}{(p^2 + 1)^2(p^2 + q^2)}. \tag{22}$$

After simple calculations we obtain

$$\int\frac{(p^2 - 1)\,\mathrm{d}p}{(p^2 + 1)(p^2 + q^2)} = \frac{1}{q^2 - 1}\left[-2\arctan p + \frac{\mathrm{i}(q^2 + 1)}{2q}\ln\left(\frac{p + \mathrm{i}q}{p - \mathrm{i}q}\right)\right] + \mathrm{const},$$
$$\int\frac{p^2\mathrm{d}p}{(p^2 + 1)^2(p^2 + q^2)} = \frac{q^2}{(q^2 - 1)^2}\left[\arctan p - \frac{\mathrm{i}}{2q}\ln\left(\frac{p + \mathrm{i}q}{p - \mathrm{i}q}\right)\right]$$
$$- \frac{1}{2(q^2 - 1)}\left(\arctan p + \frac{p}{p^2 + 1}\right) + \mathrm{const}. \tag{23}$$

From (21)–(23) we find

$$\int_{-\infty}^{\infty} \frac{\tanh s}{\cosh(s-z)}\,\mathrm{d}s = \frac{\pi(q-1)}{q+1},$$
$$\int_{-\infty}^{\infty} \frac{1}{\cosh^2 s \cosh(s-z)}\,\mathrm{d}s = \frac{2\pi q}{(q+1)^2}, \qquad q = \exp z. \tag{24}$$

Formulae (20) and (24) give

$$\varphi^a(y_0, z) = \left[k_1^a(y_0) - \mathrm{i}k_2^a(y_0)\right]\tanh(z/2). \tag{25}$$

From (17) and (25) we obtain

$$h^a(y_0, \theta, \sigma) = \mathrm{Re}\,\varphi^a(y_0, \theta + \mathrm{i}\sigma) = \frac{k_1^a(y_0)(\mathrm{e}^{2\theta} - 1) + 2k_2^a(y_0)\,\mathrm{e}^{\theta}\sin\sigma}{\mathrm{e}^{2\theta} + 2\mathrm{e}^{\theta}\cos\sigma + 1}. \tag{26}$$

Consider the non-Abelian wave propagating in the semi-infinite space $x^3 \geqslant 0$ and corresponding to the particular solution (26). Then, using formulae (18), (26) and the evident relations $\sin\sigma = (x^2/x^3)\cos\sigma$, $\cos\sigma = [1 + (x^2/x^3)^2]^{-1/2}$, where $\sigma = \arctan(x^2/x^3)$, we find after reductions

$$A^{a,0} = \frac{k_1^a(x^0 - r)x^1 + k_2^a(x^0 - r)x^2}{r + x^3}, \qquad x^3 \geqslant 0. \tag{27}$$

Formulae (3), (4) and (27) give us an example of non-Abelian expanding waves.

The found non-Abelian wave solutions could be applied to detect cosmic sources of Yang–Mills fields. Consider fields with $SU(2)$ symmetry. Then the presence of non-Abelian waves implies the fundamental possibility of electroweak interactions with them. That is why cosmic sources of Yang–Mills fields with $SU(2)$ symmetry could be detected by means of experiments with neutrinos. In particular, if the solar radiation could have non-Abelian components, then they could be revealed by observing possible changes of the energies of neutrinos during solar eclipses.

References

[1] Ryder L H 1987 *Quantum Field Theory* (Cambridge: Cambridge University Press)
[2] Faddeev L D and Slavnov A A 1990 *Gauge Fields: Introduction to Quantum Theory* (London: Benjamin)
[3] Coleman S R 1977 *Phys. Lett.* B **70** 59
[4] Campbell W B and Morgan T A 1979 *Phys. Lett.* B **84** 87
[5] Gueven R 1979 *Phys. Rev.* D **19** 471
[6] Lo S-Y, Desmond P and Kovach E 1980 *Phys. Lett.* B **90** 419
[7] Basler M and Hädicke A 1984 *Phys. Lett.* B **144** 83
[8] Oh C H and Teh R 1985 *J. Math. Phys.* **26** 841
[9] Raczka P A 1986 *Phys. Lett.* B **177** 60
[10] Blaizot J-P and Lancu E 1994 *Phys. Lett.* B **326** 138
[11] Rabinowitch A S 2006 *Theor. Math. Phys.* **148** 1081
[12] Barut A O and Raczka R 1986 *Theory of Group Representations and Applications* (Singapore: World Scientific)
[13] Lavrentiev M A and Chabat B V 1972 *Méthodes de la théorie des fonctions d'une variable complexe* (Moscow: Mir)
[14] Lawrentjew M A and Schabat B W 1967 *Methoden der Komplexen Funktionentheorie* (Berlin: VEB Deutscher Verlag der Wissenschaften)